MATH 2020A
Fall 2020

Lecture 20

- surface integral of a function
- surface integral of a vector field.

Let G be a function defined on a surface S. We define its integral to be

$$\iint_S G\,d\sigma = \iint_D G(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\,dA(u,v)$$

when $(u,v)\in D$ is a regular 1-1 onto parametrization of S.

When it is a graph $(x,y)\in D$, $(x,y,f(x,y))$, the formula becomes

$$\iint_S G\,d\sigma = \iint_D G(x,y,f(x,y))\sqrt{1+|\nabla f|^2}\,dA(x,y).$$

In case of level set, $F(x,y,z)=c$,

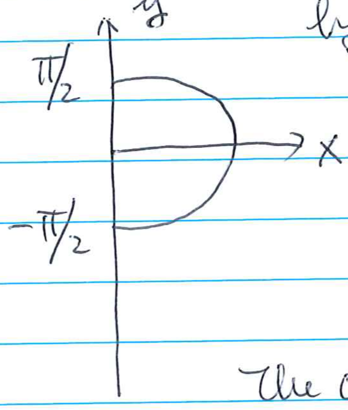
$$\iint_S G\,d\sigma = \iint_D G(x,y,z)\frac{|\nabla F|}{|F_z|}\,dA(x,y)$$

$(z=f(x,y))$.

When $G=\delta\geq 0$ density of any object occupying S,

$\iint_S G\,d\sigma$ is the mass of this object.

$\boxed{\text{e.g. 5}}$ Let S be the surface of revolution obtained by rotating $x = \cos y$, $-\pi/2 \le y \le \pi/2$. Find



$$\iint_S \sqrt{1-x^2-y^2}\, d\sigma.$$

The curve in xy-plane is $(\cos y, y)$ when y is the parameter, so S is parametrized by

$$(\alpha, y) \overset{\vec{r}}{\longmapsto} (\cos y \cos\alpha,\ \cos y \sin\alpha,\ y),$$

$$\vec{r}_\alpha = (-\cos y \sin\alpha,\ \cos y \cos\alpha,\ 0)$$

$$\vec{r}_y = (-\sin y \cos\alpha,\ -\sin y \sin\alpha,\ 1)$$

$$\vec{r}_\alpha \times \vec{r}_y = (\cos y \cos\alpha,\ \cos y \sin\alpha,\ \cos y \sin y)$$

$$|\vec{r}_\alpha \times \vec{r}_y| = \sqrt{\cos^2 y\,(1+\sin^2 y)}$$

As $x = \cos y \cos\alpha$, $y = \cos y \cos\alpha$ (this $y \neq$ the parameter y)

$$\sqrt{1-x^2-y^2} = \sqrt{1-\cos^2 y} = |\sin y|$$

$$\therefore \iint_S \sqrt{1-x^2-y^2}\, d\sigma = \int_0^{2\pi}\int_{-\pi/2}^{\pi/2} |\sin y| \sqrt{\cos^2 y\,(1+\sin^2 y)}\, dy\, d\alpha$$

$$= 2\pi \int_{-\pi/2}^{\pi/2} |\sin y| \cos y \sqrt{1+\sin^2 y}\, dy$$

$$= 4\pi \int_0^{\pi/2} \sin y \cos y \sqrt{1+\sin^2 y}\, dy$$

$$= 4\pi \int_0^1 t\sqrt{1+t^2}\, dt = \frac{4\pi}{3}\left(2\sqrt{2}-1\right) \ \#$$

Alternate approach. The curve is $\cos y - x = 0$.

$\therefore$ Implicit form $F(x,y) = \cos y - x = 0$

For S: $F(r,z) = 0$, ie $\cos z - r = 0$.

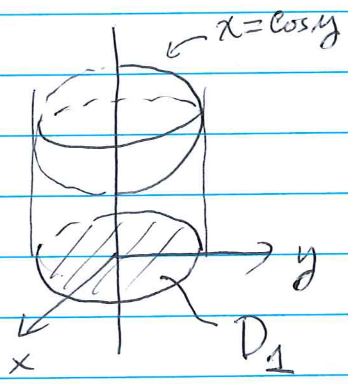
Taking square, $x^2 + y^2 - \cos^2 z = 0$

We set $G(x,y,z) = x^2 + y^2 - \cos^2 z$

and S is the level set of $G(x,y,z)$ at 0.

$$\frac{|\nabla G|}{|G_z|} = \frac{|(2x, 2y, 2\cos z \sin z)|}{|2\cos z \sin z|}$$

$$= \sqrt{\frac{4r^2 + 4r^2(1-r^2)}{4r^2(1-r^2)}} = \sqrt{\frac{2-r^2}{1-r^2}}$$



$$\therefore \iint_S \sqrt{1-x^2-y^2}\, d\sigma = 2\iint_{D_1} (1-r^2)\sqrt{\frac{2-r^2}{1-r^2}}\, dA(x,y)$$

upper and lower half

$$= 2\int_0^{2\pi}\int_0^1 \sqrt{2-r^2}\, r\,dr\,d\theta \quad \text{(switch to polar)}$$

$$= 2\pi \int_0^1 \sqrt{2-t}\, dt$$

$$= \frac{4\pi}{3}(2\sqrt{2}-1) \ \#$$

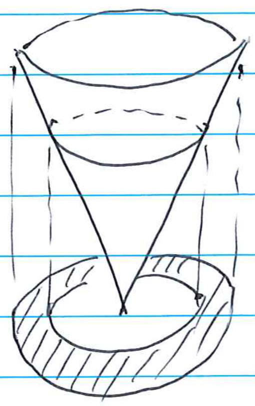
Imagine that S is a thin object with density δ, its center of mass is $(\bar{x}, \bar{y}, \bar{z})$:

$$\bar{x} = \frac{1}{M}\iint_S x\delta\, d\sigma,$$

$$\bar{y} = \frac{1}{M}\iint_S y\delta\, d\sigma,$$

$$\bar{z} = \frac{1}{M}\iint_S z\delta\, d\sigma, \quad \text{when } M = \iint_S \delta\, d\sigma.$$

$\boxed{\text{e.g. 6}}$ Find the center of mass for the ring: $z = \sqrt{x^2+y^2}$, $1 \le z \le 2$



S is given by

$\{(x, y, \sqrt{x^2+y^2}) : 1 \le x^2+y^2 \le 2\}$, and $\delta = \frac{1}{z^2}$.

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{1+f_x^2+f_y^2}$$

$$= \sqrt{1+\left(\frac{x}{r}\right)^2+\left(\frac{y}{r}\right)^2}$$

$$= \sqrt{2}$$

By symmetry $\bar{x} = \bar{y} = 0$, it suffices to find $\bar{z}$.

First,
$$M = \iint_S \frac{1}{z^2}\, d\sigma = \iint_{D_1} \frac{1}{x^2+y^2}\sqrt{2}\, dA(x,y)$$

$$= \int_0^{2\pi}\int_1^2 \frac{1}{r^2}\sqrt{2}\, r\, dr\, d\theta$$

$$= 2\pi \int_1^2 \sqrt{2}\, \frac{1}{r}\, dr$$

$$= 2\pi\sqrt{2} \log 2 .$$

always ($\log = \ln$) the natural log.

$$M\bar{z} = \iint_S \frac{1}{z^2}\, z \, d\sigma = \iint_{P_1} \frac{1}{\sqrt{x^2+y^2}} \sqrt{2}\, dA(x,y)$$

$$= \int_0^{2\pi} \int_1^2 \frac{1}{r} \sqrt{2}\, r\, dr\, d\theta$$

$$= 2\pi\sqrt{2}$$

$$\therefore (\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{2\pi\sqrt{2}}{2\pi\sqrt{2}\log 2}\right)$$

$$= \left(0, 0, \frac{1}{\log 2}\right) . \#$$

We now study the surface integral of vector field.

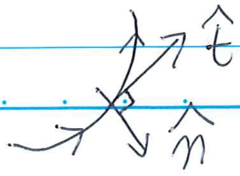
Recall for an oriented curve C, we define

$\int_C \vec{F} \cdot \hat{t}\, ds$ the circulation of $\vec{F}$ along C.

and

$\int_C \vec{F} \cdot \hat{n}\, ds$ the flux of $\vec{F}$ through C

(need an orientation on C to define $\hat{n}$ and $\hat{t}$)



Clearly, need an orientation on S too.

When S is described by a regular 1-1 onto parametrization, a natural normal $\hat{n}$ is given by

$$\hat{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}.$$

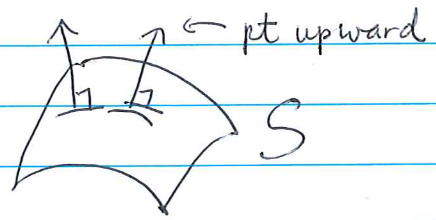
Indeed, as $\vec{r}_u$ and $\vec{r}_v$ are tangential to the surface and are linearly independent, they span the tangent plane at $\vec{r}(u,v)$. Since $\vec{r}_u \times \vec{r}_v$ are $\perp$ to $\vec{r}_u$ and $\vec{r}_v$, it is the normal direction. One may choose $\hat{n}$ or $-\hat{n}$ as the orientation of S. Here, the convention is to choose $\hat{n}$.

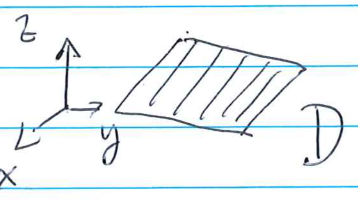
Just like the case of curves, there are 2 choice of (contin.) unit normal vector field on S.

When S is a graph $(x, y, f(x,y))$,

$$\hat{n} = \frac{\vec{r}_x \times \vec{r}_y}{|\quad|} = \frac{(-f_x, -f_y, 1)}{\sqrt{1+|\nabla f|^2}}.$$

The z-component $1/\sqrt{1+|\nabla f|^2}$ is +ve, that means the chosen $\hat{n}$ points upward.





Let S be a surface with a choice of orientation (an oriented surface) and $\vec{F}$ a v.f. on S. We define the surface integral of $\vec{F}$ over S (or the flux of $\vec{F}$ through) to be

$$\iint_S \vec{F}\cdot\hat{n}\,d\sigma$$

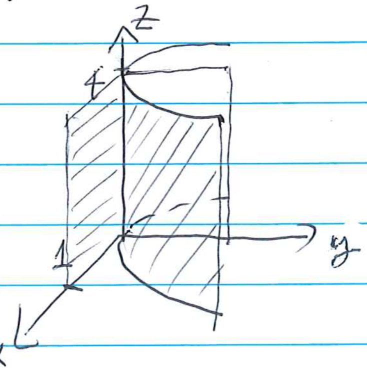
(For curves we have notation $\int_C \vec{F}\cdot d\vec{r}$, but here we don't have a common notation, one may use

$$\iint_S \vec{F}\cdot d\vec{\sigma}\,.)$$

The surface integral could be simplified to

$$\iint_D \vec{F}\cdot(\vec{r}_u\times\vec{r}_v)\,dA(u,v).$$

e.g. 7. Find the flux $\vec{F} = yz\hat{i} + x\hat{j} - z^2\hat{k}$ through the surface $y = x^2$, $0 \le x \le 1$, $0 \le z \le 4$ where $\hat{n}$ points toward the +ve x-axis.



S is the surface formed by the graph (x, x^2, z) over the rectgle $R = \{(x,z), 0 \le x \le 1, 0 \le z \le 4\}$

$\vec{r}_x = (1, 2x, 0)$

$\vec{r}_z = (0, 0, 1)$

$\vec{r}_x \times \vec{r}_z = (2x, -1, 0)$ $\quad 2x > 0$ for $x > 0$, so the normal is

$$\hat{n} = \frac{\vec{r}_x \times \vec{r}_z}{|\vec{r}_x \times \vec{r}_z|} = \frac{(2x, -1, 0)}{\sqrt{4x^2+1}}.$$

$\therefore$ the flux is

$$\iint_R (x^2 z, x, -z^2) \cdot (2x, -1, 0)\, dA(x,z)$$

$$= \iint_R (2x^3 z - x)\, dA(x,y)$$

$$= \int_0^4 \int_0^1 (2x^3 z - x)\, dx\, dz$$

$$= 2 \quad \#$$